DAY SIX

Determinants

Learning & Revision for the Day

- Determinants
- Properties of Determinants
- Cyclic Determinants
- Area of Triangle by using Determinants
- Minors and Cofactors
- Adjoint of a Matrix
- Inverse of a Matrix
- Solution of System of Linear Equations in Two and Three Variables

Determinants

Every square matrix A can be associated with a number or an expression which is called its determinant and it is denoted by det (A) or $|A|$ or Δ.

$$\text{If } A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix}, \text{ then det } (A)=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{vmatrix}$$

- If $A=\begin{bmatrix} a & b \\ c & d\end{bmatrix}$, then $|A|=\begin{vmatrix} a & b \\ c & d\end{vmatrix}=ad-bc$
- If $A=\begin{bmatrix} a & b & c \\ p & q & r \\ u & v & w\end{bmatrix}$, then $|A|=\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w\end{vmatrix}$

$$=a\begin{vmatrix} q & r \\ v & w\end{vmatrix}-b\begin{vmatrix} p & r \\ u & w\end{vmatrix}+c\begin{vmatrix} p & q \\ u & v\end{vmatrix} \quad [\text{expanding along } R_1]$$

$$=a(qw-vr)-b(pw-ur)+c(pv-uq)$$

There are six ways of expanding a determinant of order 3 corresponding to each of three rows (R_1, R_2, R_3) and three columns (C_1, C_2, C_3).

NOTE
- Rule to put + or – sign in the expansion of determinant of order 3. $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & +\end{vmatrix}$
- A square matrix A is said to be singular, if $|A|=0$ and non-singular, if $|A|\neq 0$.

Properties of Determinants

(i) If each element of a row (column) is zero, then $\Delta = 0$.

(ii) If two rows (columns) are proportional, then $\Delta = 0$.

(iii) $|A^T| = |A|$, where A^T is a transpose of a matrix.

(iv) If any two rows (columns) are interchanged, then Δ becomes $-\Delta$.

(v) If each element of a row (column) of a determinant is multiplied by a constant k, then the value of the new determinant is k times the value of the original determinant

(vi) $\det(kA) = k^n \det(A)$, if A is of order $n \times n$.

(vii) If each element of a row (column) of a determinant is written as the sum of two or more terms, then the determinant can be written as the sum of two or more determinants i.e.

$$\begin{vmatrix} a_1 + a_2 & b & c \\ p_1 + p_2 & q & r \\ u_1 + u_2 & v & w \end{vmatrix} = \begin{vmatrix} a_1 & b & c \\ p_1 & q & r \\ u_1 & v & w \end{vmatrix} + \begin{vmatrix} a_2 & b & c \\ p_2 & q & r \\ u_2 & v & w \end{vmatrix}$$

(viii) If a scalar multiple of any row (column) is added to another row (column), then Δ is unchanged

i.e. $\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix} = \begin{vmatrix} a & b & c \\ p+ka & q+kb & r+kc \\ u & v & w \end{vmatrix}$, which is obtained by the operation $R_2 \to R_2 + kR_1$

Product of Determinants

If $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $|B| = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$, then

$$|A| \times |B| = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix} = |AB|$$

[multiplying row by row]

We can multiply rows by columns or columns by rows or columns by columns

NOTE
- $|AB| = |A|\,|B| = |BA| = |A^TB| = |AB^T| = |A^TB^T|$
- $|A^n| = |A|^n, n \in Z^+$

Cyclic Determinants

In a cyclic determinant, the elements of row (or column) are arranged in a systematic order and the value of a determinant is also in systematic order.

(i) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

(ii) $\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$

(iii) $\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

(iv) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$
$= -(a^3+b^3+c^3-3abc)$

(v) $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$

Area of Triangle by using Determinants

If $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of ΔABC, then

$$\text{Area of } \Delta ABC = \frac{1}{2}\left\|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}\right\|$$

$$= \frac{1}{2}\left|[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]\right|$$

If these three points are collinear, then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$ and *vice-versa*.

Minors and Cofactors

The **minor** M_{ij} of the element a_{ij} is the determinant obtained by deleting the ith row and jth column of Δ.

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$,

then $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ etc.

The **cofactor** C_{ij} of the element a_{ij} is $(-1)^{i+j}M_{ij}$.

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then $C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, $C_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ etc.

The sum of product of the elements of any row (or column) with their corresponding cofactors is equal to the value of determinant.

i.e. $\Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$
$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$
$= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

But if elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero.

Adjoint of a Matrix

If $A=\left[a_{ij}\right]_{nxn}$, then adjoint of A, denoted by adj (A), is defined as $[C_{ij}]_{nxn}^{T}$, where C_{ij} is the cofactor of a_{ij}.

If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then adj $(A)=\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^{T}$

NOTE • If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then adj $(A)=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Properties of Adjoint of a Matrix

Let A be a square matrix of order n, then

(i) $(\text{adj } A)A = A(\text{adj } A)=|A|\cdot I_n$

(ii) $|\text{adj } A|=|A|^{n-1}$, if $|A|\neq 0$

(iii) $\text{adj }(AB)=(\text{adj } B)(\text{adj } A)$

(iv) $\text{adj}(A^T)=(\text{adj } A)^T$

(v) $\text{adj}(\text{adj } A)=|A|^{n-2}A$, if $|A|\neq 0$

(vi) $|\text{adj}(\text{adj } A)|=|A|^{(n-1)^2}$, if $|A|\neq 0$

Inverse of a Matrix

Let A be any non-singular (i.e. $|A|\neq 0$) square matrix, then inverse of A can be obtained by following two ways.

1. Using determinants

In this, $A^{-1}=\dfrac{1}{|A|}\text{adj }(A)$

2. Using Elementary operations

In this, first write $A=IA$ (for applying row operations) or $A=AI$ (for applying column operations) and then reduce A of LHS to I, by applying elementary operations simultaneously on A of LHS and I of RHS. If it reduces to $I=PA$ or $I=AP$, then $P=A^{-1}$.

Properties of Inverse of a Matrix

(i) A square matrix is invertible if and only if it is non-singular.

(ii) If $A=\text{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$, then $A^{-1}=\text{diag}(\lambda_1^{-1},\lambda_2^{-1},\ldots,\lambda_n^{-1})$ provided $\lambda_i\neq 0\,\forall,\, i=1,2,\ldots n$.

Solution of System of Linear Equations in Two and Three Variables

Let system of linear equations in three variables be

$a_1x+b_1y+c_1z=d_1,\; a_2x+b_2y+c_2z=d_2$

and $a_3x+b_3y+c_3z=d_3$.

Now, we have two methods to solve these equations.

1. Matrix Method

In this method we first write the above system of equations in matrix form as shown below.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \text{ or } AX=B$$

where, $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

***Case* I When system of equations is non-homogeneous (i.e. when $B\neq 0$).**

- If $|A|\neq 0$, then the system of equations is consistent and has a unique solution given by $X=A^{-1}B$.
- If $|A|=0$ and $(\text{adj } A)$, $B\neq 0$, then the system of equations is inconsistent and has no solution.
- If $|A|=0$ and $(\text{adj } A)\cdot B=O$, then the system of equations may be either consistent or inconsistent according as the system have infinitely many solutions or no solution.

***Case* II When system of equations is homogeneous (i.e. when $B=0$).**

- If $|A|\neq 0$, then system of equations has only trivial solution, namely $x=0$, $y=0$ and $z=0$.
- If $|A|=0$, then system of equations has non-trivial solution, which will be infinite in numbers.

2. Cramer's Rule Method

In this method we first determine

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\; D_1=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$D_2=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } D_3=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

***Case* I When system of equations is non-homogeneous**

- If $D\neq 0$, then it is consistent with unique solution given by $x=\dfrac{D_1}{D}, y=\dfrac{D_2}{D}, z=\dfrac{D_3}{D}$.
- If $D=0$ and atleast one of D_1, D_2 and D_3 is non-zero, then it is inconsistent (no solution).
- If $D=D_1=D_2=D_3=0$, then it may be consistent or inconsistent according as the system have infinitely many solutions or no solution.

Case II When system of equations is homogeneous

- If $D\neq 0$, then $x=y=z=0$ is the only solution, i.e. the trivial solution.
- If $D=0$, then it has infinitely many solutions.

Above methods can be used, in a similar way, for the solution of system of linear equations in two variables.

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 If $x = cy + bz$, $y = az + cx$ and $z = bx + ay$, where x, y and z are not all zero, then $a^2 + b^2 + c^2$ is equal to

(a) $1 + 2abc$ (b) $1 - 2abc$
(c) $1 + abc$ (d) $abc - 1$

2 Consider the set A of all determinants of order 3 with entries 0 or 1 only. Let B be the subset of A consisting of all determinants with value 1 and C be the subset of A consisting of all determinants with value –1. Then,

(a) C is empty
(b) B and C have the same number of elements
(c) $A = B \cup C$
(d) B has twice as many elements as C

3 If x, y and z are positive, then $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ is equal to

(a) 0 (b) 1 (c) –1 (d) None of these

4 If a, b and c are cube roots of unity, then

$$\begin{vmatrix} e^a & e^{2a} & e^{3a} - 1 \\ e^b & e^{2b} & e^{3b} - 1 \\ e^c & e^{2c} & e^{3c} - 1 \end{vmatrix}$$ is equal to

(a) 0 (b) e (c) e^2 (d) e^3

5 If $px^4 + qx^3 + rx^2 + sx + t$

$$= \begin{vmatrix} x^2 + 3x & x - 1 & x + 3 \\ x + 1 & -2x & x - 4 \\ x - 3 & x + 4 & 3x \end{vmatrix}$$, where p, q, r, s

and t are constants, then t is equal to

(a) 0 (b) 1 (c) 2 (d) –1

6 If $f(x) = \begin{vmatrix} 1 & x & x + 1 \\ 2x & x(x - 1) & (x + 1)x \\ 3x(x - 1) & x(x - 1)(x - 2) & (x + 1)x(x - 1) \end{vmatrix}$,

then $f(50)$ is equal to

(a) 0 (b) 50 (c) 1 (d) –50

7 If α, β and γ are the roots of the equation $x^3 + px + q = 0$, then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ is

(a) 0 (b) –2 (c) 2 (d) 4

8 If ω is a cube root of unity, then a root of the following

$$\begin{vmatrix} x - \omega - \omega^2 & \omega & \omega^2 \\ \omega & x - \omega - 1 & 1 \\ \omega^2 & 1 & x - 1 - \omega^2 \end{vmatrix} = 0$$ is

(a) $x = 0$ (b) $x = -1$
(c) $x = \omega$ (d) None of these

9 If ω is an imaginary cube root of unity, then the value of

$$\begin{vmatrix} a & b\omega^2 & a\omega \\ b\omega & c & b\omega^2 \\ c\omega^2 & a\omega & c \end{vmatrix}$$ is

(a) $a^3 + b^3 + c^2 - 3abc$ (b) $a^2b - b^2c$
(c) 0 (d) $a^2 + b^2 + c^2$

10 If $\begin{vmatrix} x - 4 & 2x & 2x \\ 2x & x - 4 & 2x \\ 2x & 2x & x - 4 \end{vmatrix} = (A + Bx)(x - A)^2$, then the ordered pair (A, B) is equal to → **JEE Mains 2018**

(a) (–4,–5) (b) (–4,3)
(c) (–4,5) (d) (4,5)

11 If x, y, z are non-zero real numbers and

$$\begin{vmatrix} 1 + x & 1 & 1 \\ 1 + y & 1 + 2y & 1 \\ 1 + z & 1 + z & 1 + 3z \end{vmatrix} = 0,$$

then $x^{-1} + y^{-1} + z^{-1}$ is equal to

(a) 0 (b) –1
(c) –3 (d) –6

12 If $\begin{vmatrix} b + c & c + a & a + b \\ c + a & a + b & b + c \\ a + b & b + c & c + a \end{vmatrix} = k \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$, then k is equal to

(a) 0 (b) 1 (c) 2 (d) 3

13 Let a, b and c be such that $(b + c) \neq 0$. If

$$\begin{vmatrix} a & a + 1 & a - 1 \\ -b & b + 1 & b - 1 \\ c & c - 1 & c + 1 \end{vmatrix} + \begin{vmatrix} a + 1 & b + 1 & c - 1 \\ a - 1 & b - 1 & c + 1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix} = 0,$$

then the value of n is

(a) zero (b) an even integer
(c) an odd integer (d) an integer

14 If one of the roots of the equation

$$\begin{vmatrix} 7 & 6 & x^2 - 13 \\ 2 & x^2 - 13 & 2 \\ x^2 - 13 & 3 & 7 \end{vmatrix} = 0$$ is $x = 2$, then sum of all other five roots is

(a) – 2 (b) 0
(c) $2\sqrt{5}$ (d) $\sqrt{15}$

15 If a, b and c are sides of a scalene triangle, then the value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is → **JEE Mains 2013**

(a) non-negative (b) negative
(c) positive (d) non-positive

16 If the system of linear equations

$$x + ky + 3z = 0,\ 3x + ky - 2z = 0$$

and $\quad 2x + 4y - 3z = 0$

has a non-zero solution (x, y, z), then $\frac{xz}{y^2}$ is equal to

→ **JEE Mains 2018**

(a) -10 (b) 10
(c) -30 (d) 30

17 If $A = [a_{ij}]_{nxn}$ and $a_{ij} = (i^2 + j^2 - ij)(j - i)$, n is odd, then which of the following is not the value of tr (A).

(a) 0 (b) $|A|$
(c) $2|A|$ (d) None of these

18 If the coordinates of the vertices of an equilateral triangle with sides of length a are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3),

then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$ is equal to

→ **NCERT Exemplar**

(a) $\frac{a^4}{4}$ (b) $\frac{3a^2}{4}$ (c) $\frac{5a^4}{4}$ (d) $\frac{3a^4}{4}$

19 The points $A(a, b + c), B(b, c + a)$ and $C(c, a + b)$ are

(a) vertices of an isosceles triangle
(b) vertices of an equilateral triangle
(c) collinear
(d) None of the above

20 If A_1, B_1, C_1, ... are respectively the co-factors of the elements $x_1, y_1, z_1, \ldots$ of the determinant

$$\Delta = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}, \text{ then } \begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix} \text{ equals to}$$

(a) $x_1\Delta$ (b) $x_1x_3\Delta$
(c) $(x_1 + y_1)\Delta$ (d) None of these

21 If $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$, then adj $(3A^2 + 12A)$ is equal to

→ **JEE Mains 2017**

(a) $\begin{bmatrix} 72 & -84 \\ -63 & 51 \end{bmatrix}$ (b) $\begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$

(c) $\begin{bmatrix} 51 & 84 \\ 63 & 72 \end{bmatrix}$ (d) $\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$

22 Which of the following is/are incorrect?

(i) Adjoint of a symmetric matrix is symmetric
(ii) Adjoint of a unit matrix is a unit matrix
(iii) $A(adj\ A) = (adj\ A) = |A| I$
(iv) Adjoint of a diagonal matrix is a diagonal matrix .

(a) (i) (b) (ii)
(c) (iii) and (iv) (d) None of these

23 If $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3×3 matrix A and $|A| = 4$, then α is equal to

→ **JEE Mains 2013**

(a) 4 (b) 11 (c) 5 (d) 0

24 If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and $A\ adj\ A = AA^T$, then $5a + b$ is equal to

→ **JEE Mains 2016**

(a) -1 (b) 5 (c) 4 (d) 13

25 If $A = \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}, B = \begin{bmatrix} q & -b & y \\ -p & a & -x \\ r & -c & z \end{bmatrix}$ and if A is invertible, then which of the following is not true?

(a) $|A| = -|B|$ and $|adj\ A| \neq |adj\ B|$
(b) $|A| = -|B|$ and $|adj\ A| = |adj\ B|$
(c) A is invertible iff B is invertible
(b) None of the above

26 If A an 3×3 non-singular matrix such that $AA' = A'A$ and $B = A^{-1}A'$, then BB' is equal to

→ **JEE Mains 2014**

(a) $I + B$ (b) I (c) B^{-1} (d) $(B^{-1})'$

27 If $\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, then

(a) $a = 1, b = 1$
(b) $a = \sin 2\theta, b = \cos 2\theta$
(c) $a = \cos 2\theta, b = \sin 2\theta$
(d) None of the above

28 If $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ and u_1, u_2 are column matrices such that $Au_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $Au_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, then $u_1 + u_2$ is equal to

(a) $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

29 The number of values of k, for which the system of equations $(k+1)x + 8y = 4k$ and $kx + (k+3)y = 3k - 1$ has no solution, is

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(a) infinite (b) 1
(c) 2 (d) 3

30 The system of equations

$$\alpha x + y + z = \alpha - 1,\ x + \alpha y + z = \alpha - 1$$

and $\quad x + y + \alpha z = \alpha - 1$

has no solution, if α is

(a) 1 (b) not -2
(c) either -2 or 1 (d) -2

31 If the system of linear equations

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$$x_1 + 2x_2 + 3x_3 = 6,\ x_1 + 3x_2 + 5x_3 = 9$$
$$2x_1 + 5x_2 + ax_3 = b$$

is consistent and has infinite number of solutions, then

(a) $a = 8$, b can be any real number
(b) $b = 15$, a can be any real number
(c) $a \in R - \{8\}$ and $b \in R - \{15\}$
(d) $a = 8, b = 15$

32 If the trivial solution is the only solution of the system of equations

$$x - ky + z = 0, \; kx + 3y - kz = 0$$

and $3x + y - z = 0$

Then, the set of all values of k is

(a) $\{2, -3\}$ (b) $R - \{2, -3\}$ (c) $R - \{2\}$ (d) $R - \{-3\}$

33 Let A, other than I or $-I$, be a 2×2 real matrix such that $A^2 = I$, I being the unit matrix. Let tr (A) be the sum of diagonal elements of A. → **JEE Mains 2013**

Statement I tr $(A) = 0$

Statement II det $(A) = -1$

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

34 The set of all values of λ for which the system of linear equations $2x_1 - 2x_2 + x_3 = \lambda x_1$, $2x_1 - 3x_2 + 2x_3 = \lambda x_2$ and $-x_1 + 2x_2 = \lambda x_3$ has a non-trivial solution.

(a) is an empty set → **JEE Mains 2015**
(b) is a singleton set
(c) contains two elements
(d) contains more than two elements

35 **Statement I** Determinant of a skew-symmetric matrix of order 3 is zero.

Statement II For any matrix A, $\det(A^T) = \det(A)$ and $\det(-A) = -\det(A)$.

Where, $\det(A)$ denotes the determinant of matrix A. Then, → **JEE Mains 2013**

(a) Statement I is true and Statement II is false
(b) Both statements are true
(c) Both statements are false
(d) Statement I is false and Statement II is true

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 If $a^2 + b^2 + c^2 = -2$ and

$$f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix},$$

then $f(x)$ is a polynomial of degree.

(a) 2 (b) 3 (c) 0 (d) 1

2 If A is a square matrix of order 3 such that $|A| = 2$, then $|(\text{adj } A^{-1})^{-1}|$ is

(a) 1 (b) 2 (c) 4 (d) 8

3 The equations $(k-1)x + (3k+1)y + 2kz = 0$,

$$(k-1)x + (4k-2)y + (k+3)z = 0$$

and $2x + (3k+1)y + 3(k-1)z = 0$

gives non-trivial solution for some values of k, then the ratio $x : y : z$ when k has the smallest of these values.

(a) 3:2:1 (b) 3:3:2 (c) 1:3:1 (d) 1:1:1

4 If $x = 1 + 2 + 4 + \dots$ upto k terms, $y = 1 + 3 + 9 + \dots$ upto k terms and $c = 1 + 5 + 25 + \dots$ upto k terms. Then,

$$\Delta = \begin{vmatrix} x & 2y & 4z \\ 3 & 3 & 3 \\ 2^k & 3^k & 5^k \end{vmatrix} \text{ equals to}$$

(a) $(20)^k$ (b) 5^k (c) 0 (d) $2^k + 3^k + 5^k$

5 Product of roots of equation $\begin{vmatrix} 1+2x & 1 & 1-x \\ 2-x & 2+x & 3+x \\ x & 1+x & 1-x^2 \end{vmatrix} = 0$

(a) $\frac{1}{2}$ (b) $\frac{3}{4}$ (c) $\frac{4}{3}$ (d) $\frac{1}{4}$

6 If the equations $a(y+z) = x$, $b(z+x) = y$, $c(x+y) = z$ have non-trivial solution, then $\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}$ is equal to

(a) 1 (b) 2 (c) -1 (d) -2

7 Let a, b and c be positive real numbers. The following system of equations in x, y and z.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ and } -\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

has

(a) no solution (b) unique solution
(c) infinitely many solutions (d) finitely many solutions

8 If S is the set of distinct values of b for which the following system of linear equations

$$x + y + z = 1, \; x + ay + z = 1 \text{ and } ax + by + z = 0$$

has no solution, then S is → **JEE Mains 2017**

(a) an infinite set
(b) a finite set containing two or more elements
(c) singleton set
(d) a empty set

9 If M is a 3×3 matrix, where $M^T M = I$ and $\det(M) = 1$, then the value of det $(M - I)$ is

(a) -1 (b) 1
(c) 0 (d) None of these

10 If $a_1, a_2, \dots, a_n, \dots$ are in GP, then the determinant

$$\Delta = \begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix} \text{ is equal to}$$

(a) 2 (b) 4 (c) 0 (d) 1

11 Let A be a square matrix of order 2 with $|A| \neq 0$ such that $|A + |A| \operatorname{adj}(A)| = 0$, then the value of $|A - |A| \operatorname{adj}(A)|$ is

(a) 1 (b) 2
(c) 3 (d) 4

12 Let P and Q be 3×3 matrices $P \neq Q$. If $P^3 = Q^3$ and $P^2Q = Q^2P$, then the determinant of $(P^2 + Q^2)$ is

(a) -2 (b) 1
(c) 0 (d) -1

13 If $\alpha, \beta \neq 0$, $f(n) = \alpha^n + \beta^n$ and $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$

$= K(1-\alpha)^2 (1-\beta)^2 (\alpha - \beta)^2$, then K is equal to

➡ **JEE Mains 2014**

(a) $\alpha\beta$ (b) $\frac{1}{\alpha\beta}$ (c) 1 (d) -1

14 Area of triangle whose vertices are $(a, a^2), (b, b^2), (c, c^2)$ is $\frac{1}{2}$, and the area of triangle whose vertices are $(p, p^2), (q, q^2)$ and (r, r^2) is 4, then the value of

$$\begin{vmatrix} (1+ap)^2 & (1+bp)^2 & (1+cp)^2 \\ (1+aq)^2 & (1+bq)^2 & (1+cq)^2 \\ (1+ar)^2 & (1+br)^2 & (1+cr)^2 \end{vmatrix} \text{ is}$$

(a) 2 (b) 4 (c) 8 (d) 16

15 Let $\det A = \begin{vmatrix} l & m & n \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ and if $(l-m)^2 + (p-q)^2 = 9$, $(m-n)^2 + (q-r)^2 = 16$, $(n-l)^2 + (r-p)^2 = 25$, then the value of $(\det A)^2$ equals to

(a) 36 (b) 100 (c) 144 (b) 169

ANSWERS

SESSION 1

1 (b)	**2** (b)	**3** (a)	**4** (a)	**5** (a)	**6** (a)	**7** (a)	**8** (a)	**9** (c)	**10** (c)
11 (c)	**12** (c)	**13** (c)	**14** (a)	**15** (b)	**16** (b)	**17** (d)	**18** (d)	**19** (c)	**20** (a)
21 (b)	**22** (d)	**23** (b)	**24** (b)	**25** (a)	**26** (b)	**27** (c)	**28** (d)	**29** (d)	**30** (d)
31 (d)	**32** (b)	**33** (b)	**34** (c)	**35** (a)					

SESSION 2

1 (a)	**2** (c)	**3** (d)	**4** (c)	**5** (a)	**6** (b)	**7** (d)	**8** (c)	**9** (c)	**10** (c)
11 (d)	**12** (c)	**13** (c)	**14** (d)	**15** (c)					

Hints and Explanations

SESSION 1

1 Given equation can be rewritteen as

$$x - cy - bz = 0$$
$$cx - y + az = 0$$
$$bx + ay - z = 0$$

Since, x, y and z are not all zero

$\therefore$ The above system have non-trivial solution.

$$\therefore \begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$$\Rightarrow 1(1 - a^2) + c(-c - ab) - b(ac + b) = 0$$
$$\Rightarrow \quad 1 - a^2 - c^2 - abc - abc - b^2 = 0$$
$$\Rightarrow \quad a^2 + b^2 + c^2 = 1 - 2abc$$

2 If we interchange any two rows of a determinant in the set B, its value becomes -1 and hence it is in C. Likewise, for every determinant in C, there is corresponding determinant in B. So , B and C have the same number of elements.

3 Let $\Delta = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$

$$= \begin{vmatrix} \frac{\log x}{\log x} & \frac{\log y}{\log x} & \frac{\log z}{\log x} \\ \frac{\log x}{\log y} & \frac{\log y}{\log y} & \frac{\log z}{\log y} \\ \frac{\log x}{\log z} & \frac{\log y}{\log z} & \frac{\log z}{\log z} \end{vmatrix}$$

By taking common factors from the rows, we get

$$\Delta = \frac{1}{\log x \cdot \log y \cdot \log z} \begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix}$$

Now, by taking common factor from the columns, we get $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$

4 Let $\Delta = \begin{vmatrix} e^a & e^{2a} & e^{3a} \\ e^b & e^{2b} & e^{3b} \\ e^c & e^{2c} & e^{3c} \end{vmatrix} - \begin{vmatrix} e^a & e^{2a} & 1 \\ e^b & e^{2b} & 1 \\ e^c & e^{2c} & 1 \end{vmatrix}$

$$= e^a \cdot e^b \cdot e^c \begin{vmatrix} 1 & e^a & e^{2a} \\ 1 & e^b & e^{2b} \\ 1 & e^c & e^{2c} \end{vmatrix} + \begin{vmatrix} e^a & 1 & e^{2a} \\ e^b & 1 & e^{2b} \\ e^c & 1 & e^{2c} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & e^a & e^{2a} \\ 1 & e^b & e^{2b} \\ 1 & e^c & e^{2c} \end{vmatrix} - \begin{vmatrix} 1 & e^a & e^{2a} \\ 1 & e^b & e^{2b} \\ 1 & e^c & e^{2c} \end{vmatrix}$$

$= 0$ $[\because a + b + c = 0 \Rightarrow e^{a+b+c} = 1]$

5 Put $x = 0$ in the given equation, we get

$$t = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix} = -12 + 12 = 0$$

6 On taking common factors x from C_2, $(x + 1)$ from C_3 and $(x - 1)$ from R_3, we get

$$f(x) = x(x^2 - 1) \begin{vmatrix} 1 & 1 & 1 \\ 2x & x - 1 & x \\ 3x & x - 2 & x \end{vmatrix}$$

$$= x(x^2 - 1) \begin{vmatrix} 1 & 0 & 0 \\ 2x & -(x + 1) & 1 \\ 3x & -2(x + 1) & 2 \end{vmatrix}$$

$$= 0 \quad \begin{bmatrix} C_3 \to C_3 - C_2, \\ C_2 \to C_2 - C_1 \end{bmatrix}$$

$\therefore \quad f(50) = 0$

7 Clearly, $\alpha + \beta + \gamma = 0$

On applying $C_1 \to C_1 + C_2 + C_3$ in the given determinant, we get

$$\therefore \Delta = \begin{vmatrix} \alpha + \beta + \gamma & \beta & \gamma \\ \alpha + \beta + \gamma & \gamma & \alpha \\ \alpha + \beta + \gamma & \alpha & \beta \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \beta & \gamma \\ 0 & \gamma & \alpha \\ 0 & \alpha & \beta \end{vmatrix} = 0$$

8 On applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} x & \omega & \omega^2 \\ x & x + \omega^2 & 1 \\ x & 1 & x + \omega \end{vmatrix} = 0$$

$[\because 1 + \omega + \omega^2 = 0]$

$$\Rightarrow \quad x \begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & x + \omega^2 & 1 \\ 1 & 1 & x + \omega \end{vmatrix} = 0$$

$\therefore \quad x = 0$

9 $\Delta = \begin{vmatrix} a(1 + \omega) & b\omega^2 & a\omega \\ b(\omega + \omega^2) & c & b\omega^2 \\ c(\omega^2 + 1) & a\omega & c \end{vmatrix}$

$[\because C_1 \to C_1 + C_3]$

$$= \begin{vmatrix} -a\omega^2 & b\omega^2 & a\omega \\ -b & c & b\omega^2 \\ -c\omega & a\omega & c \end{vmatrix}$$

$$= \omega^2 \cdot \omega \begin{vmatrix} -a & b & a\omega^2 \\ -b & c & b\omega^2 \\ -c & a & c\omega^2 \end{vmatrix}$$

$$= -\omega^5 \begin{vmatrix} a & b & a \\ b & c & b \\ c & a & c \end{vmatrix} = 0$$

10 Given, $\begin{vmatrix} x - 4 & 2x & 2x \\ 2x & x - 4 & 2x \\ 2x & 2x & x - 4 \end{vmatrix}$

$$= (A + Bx)(x - A)^2$$

On applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 5x - 4 & 2x & 2x \\ 5x - 4 & x - 4 & 2x \\ 5x - 4 & 2x & x - 4 \end{vmatrix} = (A + Bx)(x - A)^2$$

On taking common $(5x - 4)$ from C_1, we get

$$(5x - 4) \begin{vmatrix} 1 & 2x & 2x \\ 1 & x - 4 & 2x \\ 1 & 2x & x - 4 \end{vmatrix} = (A + Bx)(x - A)^2$$

On applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\therefore (5x - 4) \begin{vmatrix} 1 & 2x & 2x \\ 0 & -x - 4 & 0 \\ 0 & 0 & -x - 4 \end{vmatrix} = (A + Bx)(x - A)^2$$

On expanding along C_1, we get

$$(5x - 4)(x + 4)^2 = (A + Bx)(x - A)^2$$

On comparing, we get

$$A = -4 \text{ and } B = 5$$

11 Let $\Delta = \begin{vmatrix} 1 + x & 1 & 1 \\ 1 + y & 1 + 2y & 1 \\ 1 + z & 1 + z & 1 + 3z \end{vmatrix} = 0$

On applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we get

$$\Delta = \begin{vmatrix} x & 0 & 1 \\ y & 2y & 1 \\ -2z & -2z & 1 + 3z \end{vmatrix}$$

On expanding along R_1, we get

$$\Delta = x[2y(1 + 3z) + 2z]$$
$$+1[-2yz + 4yz] = 0$$

$$\Rightarrow \quad 2[xy + 3xyz + xz] + 2yz = 0$$
$$\Rightarrow \quad xy + yz + zx + 3xyz = 0$$
$$\Rightarrow \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -3$$

12 Let $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$

$$= \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix}$$

[using $C_1 \to C_1 + C_2 + C_3$]

$$= 2\begin{vmatrix} a+b+c & c+a & a+b \\ a+b+c & a+b & b+c \\ a+b+c & b+c & c+a \end{vmatrix}$$

[taking common 2 from C_1]

$$= 2\begin{vmatrix} a+b+c & -b & -c \\ a+b+c & -c & -a \\ a+b+c & -a & -b \end{vmatrix}$$

[using $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$]

$$= 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

[using $C_1 \to C_1 + C_2 + C_3$]

On comparing, $k = 2$

13 $\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ a & -b & c \end{vmatrix}$

$$= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & a-1 & a \\ b+1 & b-1 & -b \\ c-1 & c+1 & c \end{vmatrix}$$

[$\because |A'| = |A|$]

$$= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^{n+1} \begin{vmatrix} a+1 & a & a-1 \\ b+1 & -b & b-1 \\ c-1 & c & c+1 \end{vmatrix}$$

$$= [1 + (-1)^{n+2}] \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix}$$

This is equal to zero only, if $n+2$ is an odd i.e. n is an odd integer.

14 $\begin{vmatrix} 7 & 6 & x^2-13 \\ 2 & x^2-13 & 2 \\ x^2-13 & 3 & 7 \end{vmatrix} = 0$

On applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} x^2-4 & x^2-4 & x^2-4 \\ 2 & x^2-13 & 2 \\ x^2-13 & 3 & 7 \end{vmatrix} = 0$$

$$\Rightarrow (x^2-4)\begin{vmatrix} 1 & 1 & 1 \\ 2 & x^2-13 & 2 \\ x^2-13 & 3 & 7 \end{vmatrix} = 0$$

Now, on applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$(x^2-4)\begin{vmatrix} 1 & 0 & 0 \\ 2 & x^2-15 & 0 \\ x^2-13 & 16-x^2 & 20-x^2 \end{vmatrix} = 0$$

On expanding along R_1, we get

$$(x^2-4)(x^2-15)(x^2-20) = 0$$

Thus, other five roots are

$$-2, \pm\sqrt{15}, \pm 2\sqrt{5}$$

Hence, sum of other five roots is -2.

15 Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$= a(bc - a^2) - b(b^2 - ac) + c(ab - c^2)$
$= abc - a^3 - b^3 + abc + abc - c^3$
$= -(a^3 + b^3 + c^3 - 3abc)$
$= -(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$
$= -\frac{1}{2}(a+b+c)\{(a-b)^2 + (b-c)^2 + (c-a)^2\}$
< 0, where $a \ne b \ne c$

16 We have,

$x + ky + 3z = 0$; $3x + ky - 2z = 0$;
$2x + 4y - 3z = 0$

System of equation has non-zero solution, if

$$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 4 & -3 \end{vmatrix} = 0$$

$\Rightarrow (-3k+8) - k(-9+4) + 3(12-2k) = 0$
$\Rightarrow -3k + 8 + 9k - 4k + 36 - 6k = 0$
$\Rightarrow -4k + 44 = 0 \Rightarrow k = 11$

Let $z = \lambda$, then we get

$x + 11y + 3\lambda = 0$ ...(i)
$3x + 11y - 2\lambda = 0$ ...(ii)
and $2x + 4y - 3\lambda = 0$ ...(iii)

On solving Eqs. (i) and (ii), we get

$$x = \frac{5\lambda}{2}, y = \frac{-\lambda}{2}, z = \lambda$$

$$\Rightarrow \frac{xz}{y^2} = \frac{5\lambda^2}{2 \times \left(-\frac{\lambda}{2}\right)^2} = 10$$

17 We have,

$a_{ij} = (i^2 + j^2 - ij)(j - i)$
$\therefore a_{ji} = (i^2 + j^2 - ij)(i - j)$
$= -(i^2 + j^2 - ij)(j - i) = -a_{ij}$

$\Rightarrow$ A is a skew-symmetric matrix of odd order.

$\therefore$ tr$(A) = 0$ and $|A| = 0$

18 If $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are the vertices of a triangle, then Area

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{...(i)}$$

Also, we know that, if a is the length of an equilateral triangle, then

$$\text{Area} = \frac{\sqrt{3}}{4}a^2 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{\sqrt{3}}{4}a^2 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Rightarrow \frac{\sqrt{3}}{2}a^2 = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

On squaring both sides, we get

$$\frac{3}{4}a^4 = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$$

19 Clearly, area of

$$(\Delta ABC) = \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

$$= \frac{1}{2}(a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix} = 0$$

[$\because C_2 \to C_2 + C_1$]

20 Clearly, $B_2 = \begin{vmatrix} x_1 & z_1 \\ x_3 & z_3 \end{vmatrix} = x_1 z_3 - x_3 z_1$

$C_2 = -\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} = -(x_1 y_3 - x_3 y_1)$

$B_3 = -\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} = -(x_1 z_2 - x_2 z_1)$

and $C_3 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_1 y_2 - y_1 x_2$

$$\therefore \begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 z_3 - x_3 z_1 & -(x_1 y_3 - y_1 x_3) \\ -(x_1 z_2 - x_2 z_1) & x_1 y_2 - x_2 y_1 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 z_3 & -x_1 y_3 \\ -x_1 z_2 + x_2 z_1 & x_1 y_2 - x_2 y_1 \end{vmatrix} + \begin{vmatrix} -x_3 z_1 & y_1 x_3 \\ -x_1 z_2 + x_2 z_1 & x_1 y_2 - x_2 y_1 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 z_3 & -x_1 y_3 \\ -x_1 z_2 & x_1 y_2 \end{vmatrix} + \begin{vmatrix} x_1 z_3 & -x_1 y_3 \\ x_2 z_1 & -x_2 y_1 \end{vmatrix} + \begin{vmatrix} -x_3 z_1 & y_1 x_3 \\ -x_1 z_2 & x_1 y_2 \end{vmatrix} + \begin{vmatrix} -x_3 z_1 & y_1 x_3 \\ x_2 z_1 & -x_2 y_1 \end{vmatrix}$$

$= x_1^2(z_3 y_2 - z_2 y_3) - x_1 x_2(z_3 y_1 - z_1 y_3)$
$- x_1 x_3(z_1 y_2 - z_2 y_1) + x_2 x_3(z_1 y_1 - z_1 y_1)$
$= x_1[x_1(z_3 y_2 - z_2 y_3) - x_2(z_3 y_1 - z_1 y_3) + x_3(z_2 y_1 - z_1 y_2)]$
$= x_1 \Delta$

21 We have, $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$

$\therefore \ A^2 = A \cdot A$

$$= \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+12 & -6-3 \\ -8-4 & 12+1 \end{bmatrix} = \begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix}$$

Now, $3A^2 + 12A = 3\begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix} + 12\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 48 & -27 \\ -36 & 39 \end{bmatrix} + \begin{bmatrix} 24 & -36 \\ -48 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$$

$\therefore \text{adj}\,(3A^2 + 12A) = \begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$

22 All the given statements are true.

23 Given, adj $A = P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$

Find the determinant of P and apply the formula

$$|\text{adj}\,A| = |A|^{n-1}$$

24 Given, $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$

and A adj $A = AA^T$

Clearly, $A(\text{adj}\,A) = |A| I_2$

[$\because$ if A is square matrix of order n, then $A(\text{adj}\,A) = (\text{adj}\,A)\cdot A = |A| I_n$]

$$= \begin{vmatrix} 5a & -b \\ 3 & 2 \end{vmatrix} I_2$$

$$= (10a + 3b)\, I_2$$

$$= (10a + 3b)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10a + 3b & 0 \\ 0 & 10a + 3b \end{bmatrix} \quad \text{...(i)}$$

and $AA^T = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix}$

$$= \begin{bmatrix} 25a^2 + b^2 & 15a - 2b \\ 15a - 2b & 13 \end{bmatrix} \quad \text{...(ii)}$$

$\because \ A(\text{adj}\,A) = AA^T$

$\therefore \begin{bmatrix} 10a + 3b & 0 \\ 0 & 10a + 3b \end{bmatrix}$

$$= \begin{bmatrix} 25a^2 + b^2 & 15a - 2b \\ 15a - 2b & 13 \end{bmatrix}$$

[using Eqs. (i) and (ii)]

$\Rightarrow \quad 15a - 2b = 0$

$\Rightarrow \quad a = \dfrac{2b}{15} \quad \text{...(iii)}$

and $\quad 10a + 3b = 13 \quad \text{...(iv)}$

On substituting the value of 'a' from Eq. (iii) in Eq. (iv), we get

$$10\left(\frac{2b}{15}\right) + 3b = 13$$

$\Rightarrow \quad \dfrac{20b + 45b}{15} = 13$

$\Rightarrow \quad \dfrac{65b}{15} = 13 \Rightarrow b = 3$

Now, substituting the value of b in Eq. (iii), we get $5a = 2$

Hence, $5a + b = 2 + 3 = 5$

25 Clearly, $|B| = \begin{vmatrix} q & -b & y \\ -p & a & -x \\ r & -c & z \end{vmatrix}$

$$= -\begin{vmatrix} q & -b & y \\ p & -a & x \\ r & -c & z \end{vmatrix}$$

(taken (-1) common from R_2)

$$= (+)\begin{vmatrix} q & b & y \\ p & a & x \\ r & c & z \end{vmatrix}$$

(taken (-1) common from C_2)

$$= -\begin{vmatrix} p & a & x \\ q & b & y \\ r & c & z \end{vmatrix} \ (\because R_1 \leftrightarrow R_2)$$

$$= -\begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix}$$

$(\because C_1 \leftrightarrow C_2$ and then $C_2 \leftrightarrow C_3)$

$= -|A^T| = -|A|$

Thus, $|A| = -|B|$

Hence, $|A| \neq 0$ iff $|B| \neq 0$

$\therefore A$ is invertible iff B is invertible

Also, $|\text{adj}\,A| = |A|^2 = |B|^2 = |\text{adj}\,B|$

26 Given, A is non-singular matrix

$\Rightarrow |A| \neq 0$.

Also we have, $AA^T = A^TA$ and $B = A^{-1}A^T$

Now, $BB' = (A^{-1}A^T)(A^{-1}A^T)^T$

$= A^{-1}A^T\, A\,(A^{-1})^T \quad [\because (A')' = A]$

$= A^{-1}AA^T\,(A^{-1})^T [\because AA^T = A^TA]$

$= IA^T\,(A^{-1})^T = A^T\,(A^{-1})^T$

$= (A^{-1}A)^T\ [\because (AB)^T = B^TA^T]$

$= I^T = I$

27 We have,

$$\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \cdot \frac{1}{1 + \tan^2\theta}\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\Rightarrow \frac{1}{1 + \tan^2\theta}\begin{bmatrix} 1 - \tan^2\theta & -2\tan\theta \\ 2\tan\theta & 1 - \tan^2\theta \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta} & \dfrac{-2\tan\theta}{1 + \tan^2\theta} \\ \dfrac{2\tan\theta}{1 + \tan^2\theta} & \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta} \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$\Rightarrow a = \cos 2\theta$ and $b = \sin 2\theta$

28 On adding Au_1 and Au_2, we get

$$Au_1 + Au_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0 \\ 0+1 \\ 0+0 \end{bmatrix}$$

$$\Rightarrow \quad A(u_1 + u_2) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Since, A is non-singular matrix, i.e. $|A| \neq 0$, on multiplying both sides by A^{-1}, we get

$$A^{-1}A(u_1 + u_2) = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow u_1 + u_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}^{-1} \times \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{...(i)}$$

Now, $\quad |A| = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix}$

$= 1\,(1 - 0) - 0 + 0 = 1$

$\therefore$ adj of A

$$= \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\therefore \ A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \quad [\because |A| = 1]$$

From Eq. (i),

$$u_1 + u_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow u_1 + u_2 = \begin{bmatrix} 1 & +0+0 \\ -2 & +1+0 \\ 1 & -2+0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

29 Given equations can be written in matrix form as $AX = B$

where, $A = \begin{bmatrix} k+1 & 8 \\ k & k+3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$

and $B = \begin{bmatrix} 4k \\ 3k - 1 \end{bmatrix}$

For no solution, $|A| = 0$

and (adj A) $B \neq 0$

Now, $|A| = \begin{vmatrix} k+1 & 8 \\ k & k+3 \end{vmatrix} = 0$

$\Rightarrow \quad (k+1)(k+3) - 8k = 0$

$\Rightarrow \quad k^2 + 4k + 3 - 8k = 0$

$\Rightarrow \quad k^2 - 4k + 3 = 0$

$\Rightarrow \quad (k-1)(k-3) = 0$

$\therefore \quad k = 1, k = 3$

Also, adj $A = \begin{bmatrix} k+3 & -8 \\ -k & k+1 \end{bmatrix}$

$\therefore (\text{adj } A)\, B = \begin{bmatrix} k+3 & -8 \\ -k & k+1 \end{bmatrix} \begin{bmatrix} 4k \\ 3k-1 \end{bmatrix}$

$= \begin{bmatrix} (k+3)(4k) - 8(3k-1) \\ -4k^2 + (k+1)(3k-1) \end{bmatrix}$

$= \begin{bmatrix} 4k^2 - 12k + 8 \\ -k^2 + 2k - 1 \end{bmatrix}$

Put $k = 1$, then

$(\text{adj } A)\, B = \begin{bmatrix} 4 - 12 + 8 \\ -1 + 2 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, not true.

Put $k = 3$, then (adj A)

$B = \begin{bmatrix} 36 - 36 + 8 \\ -9 + 6 - 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix} \neq 0$, true.

Hence, the required value of k is 3.

Alternate Method Condition for the system of equations has no solution is

$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \Rightarrow \frac{k+1}{k} = \frac{8}{k+3} \neq \frac{4k}{3k-1}$

Take $\frac{k+1}{k} = \frac{8}{k+3}$

$\Rightarrow \quad k^2 + 4k + 3 = 8k$

$\Rightarrow \quad k^2 - 4k + 3 = 0$

$\Rightarrow \quad (k-1)(k-3) = 0$

$\therefore \quad k = 1, 3$

If $k = 1$, then $\frac{8}{1+3} \neq \frac{4 \cdot 1}{2}$, [false]

and $k = 3$, then $\frac{8}{6} \neq \frac{4 \cdot 3}{9 - 1}$, [true]

$\therefore \quad k = 3$

Hence, only one value of k exists.

30 The system of given equations has no solution, if

$$\begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0$$

On applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} \alpha+2 & 1 & 1 \\ \alpha+2 & \alpha & 1 \\ \alpha+2 & 1 & \alpha \end{vmatrix} = 0$$

On applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$,

$$(\alpha + 2) \begin{vmatrix} 1 & 1 & 1 \\ 0 & \alpha - 1 & 0 \\ 0 & 0 & \alpha - 1 \end{vmatrix} = 0$$

$\Rightarrow \quad (\alpha + 2)(\alpha - 1)^2 = 0$

$\therefore \quad \alpha = 1, -2$

But $\alpha = 1$ makes given three equations same. So, the system of equations has infinite solution. Hence, answer is $\alpha = -2$ for which the system of equations has no solution.

31 For consistency $|A| = 0$ and (adj A) $B = O$

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & a \end{vmatrix} = 0$$

$\Rightarrow 1(3a - 25) - 2(a - 10) + 3(5 - 6) = 0$

$\Rightarrow \quad a = 8$

On solving, (adj A)$B = O$, we get

$b = 15$

32 Since, $x - ky + z = 0$,

$kx + 3y - kz = 0$ and

$3x + y - z = 0$ has trivial solution.

$$\therefore \begin{vmatrix} 1 & -k & 1 \\ k & 3 & -k \\ 3 & 1 & -1 \end{vmatrix} \neq 0$$

$\Rightarrow \quad 1(-3 + k) + k(-k + 3k) + 1(k - 9) \neq 0$

$\Rightarrow \quad k - 3 + 2k^2 + k - 9 \neq 0$

$\Rightarrow \quad 2k^2 + 2k - 12 \neq 0$

$\Rightarrow \quad k^2 + k - 6 \neq 0$

$\Rightarrow \quad (k+3)(k-2) \neq 0$

$\therefore \quad k \neq 2, -3$

$k \in R - \{2, -3\}$

33 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $A \neq I, -I$

and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \quad b(a + d) = 0, c(a + d) = 0$

and $\quad a^2 + bc = 1, bc + d^2 = 1$

$\Rightarrow \quad a = 1, d = -1, b = c = 0$ or

$a = -1, d = 1, b = c = 0$

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then

$\det(A) = -1$ and $\text{tr}(A) = 1 - 1 = 0$

34 Given system of linear equations are

$2x_1 - 2x_2 + x_3 = \lambda x_1$

$\Rightarrow \quad (2 - \lambda)x_1 - 2x_2 + x_3 = 0 \quad$ ...(i)

$2x_1 - 3x_2 + 2x_3 = \lambda x_2$

$\Rightarrow 2x_1 - (3 + \lambda)x_2 + 2x_3 = 0 \quad$ ...(ii)

and $\quad -x_1 + 2x_2 = \lambda x_3$

$\Rightarrow \quad -x_1 + 2x_2 - \lambda x_3 = 0 \quad$ ...(iii)

Since, the system has non-trivial solution.

$$\therefore \begin{vmatrix} 2 - \lambda & -2 & 1 \\ 2 & -(3 + \lambda) & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow (2 - \lambda)(3\lambda + \lambda^2 - 4) + 2(-2\lambda + 2) + 1(4 - 3 - \lambda) = 0$

$\Rightarrow \quad (2 - \lambda)(\lambda^2 + 3\lambda - 4) + 4(1 - \lambda) + (1 - \lambda) = 0$

$\Rightarrow (2 - \lambda)(\lambda + 4)(\lambda - 1) + 5(1 - \lambda) = 0$

$\Rightarrow \quad (\lambda - 1)[(2 - \lambda)(\lambda + 4) - 5] = 0$

$\Rightarrow \quad (\lambda - 1)(\lambda^2 + 2\lambda - 3) = 0$

$\Rightarrow \quad (\lambda - 1)[(\lambda - 1)(\lambda + 3)] = 0$

$\Rightarrow \quad (\lambda - 1)^2(\lambda + 3) = 0$

$\Rightarrow \quad \lambda = 1, 1, -3$

35 Determinant of skew-symmetric matrix of odd order is zero and of even order is perfect square.

Also, $\quad \det(A^T) = \det(A)$

and $\quad \det(-A) = (-1)^n \det(A)$

Hence, Statement II is false.

SESSION 2

1 Given that,

$$f(x) = \begin{vmatrix} 1 + a^2x & (1 + b^2)x & (1 + c^2)x \\ (1 + a^2)x & 1 + b^2x & (1 + c^2)x \\ (1 + a^2)x & (1 + b^2)x & 1 + c^2x \end{vmatrix}$$

On applying $C_1 \to C_1 + C_2 + C_3$, we get

$$f(x) = \begin{vmatrix} 1 + a^2x + x + b^2x + x + c^2x & (1 + b^2)x & (1 + c^2)x \\ x + a^2x + 1 + b^2x + x + c^2x & 1 + b^2x & (1 + c^2)x \\ x + a^2x + x + b^2x + 1 + c^2x & (1 + b^2)x & 1 + c^2x \end{vmatrix}$$

$$= \begin{vmatrix} 1 + (a^2 + b^2 + c^2 + 2)x & (1 + b^2)x & (1 + c^2)x \\ 1 + (a^2 + b^2 + c^2 + 2)x & (1 + b^2)x & (1 + c^2)x \\ 1 + (a^2 + b^2 + c^2 + 2)x & (1 + b^2)x & 1 + c^2x \end{vmatrix}$$

$$= \begin{vmatrix} 1 & (1 + b^2)x & (1 + c^2)x \\ 1 & 1 + b^2x & (1 + c^2)x \\ 1 & (1 + b^2)x & 1 + c^2x \end{vmatrix}$$

$[\because a^2 + b^2 + c^2 = -2$, given$]$

On applying

$R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$, we get

$$= \begin{vmatrix} 0 & 0 & x - 1 \\ 0 & 1 - x & x - 1 \\ 1 & (1 + b^2)x & 1 + c^2x \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x - 1 \\ 1 - x & x - 1 \end{vmatrix} = (x - 1)^2$$

Hence, $f(x)$ is of degree 2.

2 Clearly, $\left|\text{adj}\, A^{-1}\right| = |A^{-1}|^2 = \frac{1}{|A|^2}$

and $|(\text{adj}\, A^{-1})^{-1}| = \frac{1}{|(\text{adj}\, A^{-1})|}$

$= |A|^2 = 2^2 = 4$

3 For non - trivial solution,

$$\begin{vmatrix} k-1 & 3k+1 & 2k \\ k-1 & 4k-2 & k+3 \\ 2 & 3k+1 & 3(k-1) \end{vmatrix} = 0$$

$\Rightarrow \quad k = 0 \text{ or } 3$

Now, when $k = 0$, then the equation becomes

$$-x + y = 0 \quad \text{...(i)}$$

$$-x - 2y + 3z = 0 \quad \text{...(ii)}$$

and $$2x + y - 3z = 0 \quad \text{(iii)}$$

From (i), we get $x = y = \lambda$ (say), Then, from Eq. (ii), we get

$$-\lambda - 2\lambda + 3z = 0$$

$\Rightarrow \quad 3z = 3\lambda$

$\Rightarrow \quad z = \lambda$

$\therefore \quad x : y : z = 1 : 1 : 1$

4 Clearly, $x = 2^k - 1, y = \frac{3^k - 1}{2}$

and $z = \frac{5^k - 1}{4}$

$\left[\because \text{sum to } n \text{ terms of } a \text{ GP is given by } \frac{a(r^n - 1)}{r - 1}\right]$

Now, $\Delta = \begin{vmatrix} 2^k - 1 & 3^k - 1 & 5^k - 1 \\ 3 & 3 & 3 \\ 2^k & 3^k & 5^k \end{vmatrix}$

$= \begin{vmatrix} 2^k & 3^k & 5^k \\ 3 & 3 & 3 \\ 2^k & 3^k & 5^k \end{vmatrix} - 3\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2^k & 3^k & 5^k \end{vmatrix}$

$= 0 - 3 \times 0 = 0$

5 Let $p(x) = \begin{vmatrix} 1 + 2x & 1 & 1 - x \\ 2 - x & 2 + x & 3 + x \\ x & 1 + x & 1 - x^2 \end{vmatrix} = 0$

Clearly, product of roots

$= \frac{\text{Constant term}}{\text{Coefficient of } x^4}$

Here, constant term is given by

$P(0) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = -1$

and coefficient of x^4 is -2

$\therefore$ Product of roots is $\frac{1}{2}$.

6 Here, $\begin{vmatrix} -1 & a & a \\ b & -1 & b \\ c & c & -1 \end{vmatrix} = 0$

On applying, $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$\begin{vmatrix} -1 & a+1 & a+1 \\ b & -(b+1) & 0 \\ c & 0 & -(1+c) \end{vmatrix} = 0$

On applying $R_1 \to \frac{R_1}{a+1}$, $R_2 \to \frac{R_2}{b+1}$

and $R_3 \to \frac{R_3}{c+1}$, we get

$$\begin{vmatrix} -\frac{1}{a+1} & 1 & 1 \\ \frac{b}{b+1} & -1 & 0 \\ \frac{c}{c+1} & 0 & -1 \end{vmatrix} = 0$$

$\Rightarrow \quad -\frac{1}{a+1} + \frac{b}{b+1} + \frac{c}{c+1} = 0$

$\therefore -\frac{1}{a+1} + 1 - \frac{1}{b+1} + 1 - \frac{c}{c+1} = 0$

$\Rightarrow \quad \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 2$

7 Let $\frac{x^2}{a^2} = X, \frac{y^2}{b^2} = Y, \frac{z^2}{c^2} = Z$

Then, $X + Y - Z = 1, X - Y + Z = 1,$ $-X + Y + Z = 1$

Now, coefficient matrix is

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Here, $|A| \neq 0$

$\therefore$ It has unique solution, *viz.*, $X = 1$, $Y = 1$ and $Z = 1$

Hence, $x = \pm a; y = \pm b$ and $z = \pm c$.

8 Here, $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 1 \end{vmatrix}$

$= 1(a - b) - 1(1 - a) + 1(b - a^2)$

$= -(a - 1)^2$

$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 0 & b & 1 \end{vmatrix}$

$= 1(a - b) - 1(1) + 1(b) = (a - 1)$

$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a & 0 & 1 \end{vmatrix}$

$= 1(1) - 1(1 - a) + 1(0 - a) = 0$

and $\Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 0 \end{vmatrix}$

$= 1(-b) - 1(-a) + 1(b - a^2)$

$= -a(a - 1)$

For $a = 1$

$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

For $b = 1$ only

$x + y + z = 1, x + y + z = 1$

and $\quad x + y + z = 0$

i.e. no solution $\quad (\because$ RHS is not equal)

Hence, for no solution, $b = 1$ only.

9 Clearly,

$\det(M - I) = \det(M - I) \cdot \det(M)$ $\quad [\because \det(M) = 1]$

$= \det(M - I)\det(M^T)$ $\quad [\because \det(A^T) = \det(A)]$

$= \det(MM^T - M^T)$

$= \det(I - M^T) \quad [\because MM^T = I]$

$= -\det(M^T - I)$

$= -\det(M^T - I)^T$

$= -\det(M - I)$

$\Rightarrow \quad 2\det(M - I) = 0$

$\Rightarrow \quad \det(M - I) = 0$

10 Since, $a_1, a_2, \ldots, a_n, \ldots$ are in GP, then, $\log a_n, \log a_{n+1}, \log a_{n+2}, \ldots,$ $\log a_{n+8}, \ldots$are in AP.

Given that,

$$\Delta = \begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}$$

$$\therefore \Delta = \begin{vmatrix} a & a+d & a+2d \\ a+3d & a+4d & a+5d \\ a+6d & a+7d & a+8d \end{vmatrix}$$

where a and d are the first term and common difference of AP.

On applying $R_2 \to 2R_2$, we get

$$\Delta = \frac{1}{2}\begin{vmatrix} a & a+d & a+2d \\ 2a+6d & 2a+8d & 2a+10d \\ a+6d & a+7d & a+8d \end{vmatrix}$$

On applying $R_2 \to R_2 - R_3$, we get

$$\Delta = \frac{1}{2}\begin{vmatrix} a & a+d & a+2d \\ a & a+d & a+2d \\ a+6d & a+7d & a+8d \end{vmatrix} = 0$$

11 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then,

$|A| = ad - bc = k$ (say)

and $\quad \text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Now, $|A + |A|\,\text{adj}(A)| = 0$

$\Rightarrow \quad \begin{vmatrix} a + kd & (1-k)b \\ (1-k)c & d + ka \end{vmatrix} = 0$

$\Rightarrow \quad (a + kd)(d + ka) - (1-k)^2 bc$

$\Rightarrow \quad ad + a^2k + kd^2 + k^2ad$ $-(1 + k^2 - 2k)bc$

$\Rightarrow (ad - bc) + (ad - bc)k^2 + k(a^2 + d^2 + 2bc) = 0$

$$\Rightarrow (ad-bc)+(ad-bc)k^2+k(a^2+d^2)+2(ad-k))=0 \quad [\because bc = ad-k]$$

$$\Rightarrow (ad-bc)+(ad-bc)k^2+k(a+d)^2-2k^2=0$$

$$\Rightarrow (k+k^3-2k^2)+k(a+d)^2=0$$

$$\Rightarrow k[(k-1)^2+(a+d)^2]=0$$

$$\Rightarrow k=1 \text{ and } a+d=0$$

Now, $|A-|A|\operatorname{adj} A|$

$$=\begin{vmatrix} a-kd & (1+k)b \\ (1+k)c & d-ak \end{vmatrix}=\begin{vmatrix} a-d & 2b \\ 2c & d-a \end{vmatrix}$$

$$=-(a-d)^2-4bc$$

$$=-((a+d)^2-4ad)-4bc$$

$$=4ad-4bc=4k=4$$

12 On subtracting the given equation, we get

$$P^3-P^2Q=Q^3-Q^2P$$

$$\Rightarrow P^2(P-Q)=Q^2(Q-P)$$

$$\Rightarrow (P-Q)(P^2+Q^2)=0 \quad \text{...(i)}$$

Now, if $|P^2+Q^2|\neq 0$

(P^2+Q^2) is invertible .

On post multiply both sides by $(P^2+Q^2)^{-1}$, we get

$P-Q=0$, which is a contradiction.

Hence, $|P^2+Q^2|=0$

13 Let $\Delta=\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$

$$\Rightarrow \Delta=\begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$=\begin{vmatrix} 1\cdot1+1\cdot1+1\cdot1 & 1\cdot1+1\cdot\alpha+1\cdot\beta & 1\cdot1+1\cdot\alpha^2+1\cdot\beta^2 \\ 1\cdot1+\alpha\cdot1+\beta\cdot1 & 1\cdot1+\alpha\cdot\alpha+\alpha\cdot\beta & 1\cdot1+\alpha\cdot\alpha^2+\beta\cdot\beta^2 \\ 1\cdot1+1\cdot\alpha^2+1\cdot\beta^2 & 1\cdot1+\alpha^2\cdot\alpha+\beta^2\cdot\beta & 1\cdot1+\alpha^2\cdot\alpha^2+\beta^2\cdot\beta^2 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2$$

On expanding, we get

$$\Delta=(1-\alpha)^2(1-\beta)^2(\alpha-\beta^2)$$

Hence, $K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$

$$=(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$$

$\therefore \quad K=1$

14 Let $\Delta=\begin{vmatrix} (1+ap)^2 & (1+bp)^2 & (1+cp)^2 \\ (1+aq)^2 & (1+bq)^2 & (1+cq)^2 \\ (1+ar)^2 & (1+br)^2 & (1+cr)^2 \end{vmatrix}$

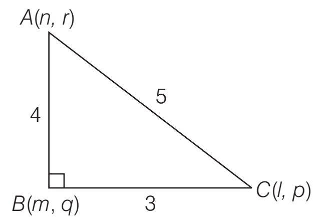
$$=\begin{vmatrix} 1+2ap+a^2p^2 & 1+2bp+b^2p^2 & 1+2cp+c^2p^2 \\ 1+2aq+a^2q^2 & 1+2bq+b^2q^2 & 1+2cq+c^2q^2 \\ 1+2ar+a^2r^2 & 1+2br+b^2r^2 & 1+2cr+c^2r^2 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & p & p^2 \\ 1 & q & q^2 \\ 1 & r & r^2 \end{vmatrix}\times\begin{vmatrix} 1 & 2a & a^2 \\ 1 & 2b & b^2 \\ 1 & 2c & c^2 \end{vmatrix}$$

$$=2\left(\begin{vmatrix} 1 & p & p^2 \\ 1 & q & q^2 \\ 1 & r & r^2 \end{vmatrix}\times\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}\right)$$

$$=2(2A_1\times 2A_2)=2(8\times1)=16$$

15 According to given conditions we get a right angled triangle whose vertices are $(n,r),(m,q)$ and (l,p).



Also, we have, $|A|=\begin{vmatrix} l & m & n \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$

$$\Rightarrow |A|^2=\begin{vmatrix} l & p & 1 \\ m & q & 1 \\ n & r & 1 \end{vmatrix}^2=[2ar(\Delta ABC)]^2$$

$$=\left[2\times\frac{1}{2}\times3\times4\right]^2=144$$